2/27/26

No office hours Mon 3/2 (candidacy exam)

The kernel of a lin. transf.

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a lin. transf. Then

$$\ker(T) = \{ \vec{x} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{0} \}$$

!! Equiv: if T is def by an $n \times m$ matrix A then

$$\ker(T) = \{ x \in \mathbb{R}^m \mid A\vec{x} = \vec{0} \}$$

$=$ solution set of the homog. linear system $A\vec{x} = \vec{0}$

Theorem Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a lin. transf. Then

(a) in $\mathbb{R}^m$, $\vec{0} \in \ker(T)$

(b) $\ker(T)$ is closed under addition of vectors

(c) $\ker(T)$ " " " scalar mult.

The pf is one of your HW problems.

Ex If A is an invertible $n \times n$ matrix, so it defines $T_A : \mathbb{R}^n \to \mathbb{R}^n$, then $\ker(T_A) = \ker(A) = \{\vec{0}\}$. This is b/c being invertible is equiv. to $A\vec{x} = \vec{0}$ having only the trivial solution. In fact:

See Summary 3.1.8 for a list of several equiv. conditions for a square matrix A to be invertible. You should know this.

Generalize this:

Theorem Suppose A is an $n \times m$ matrix with $n \neq m$. (so not square.)

(a) if $\text{rank}(A) = m$ then $\ker(A) = \{\vec{0}\}$

(b) Conversely, if $\ker(A) = \{\vec{0}\}$ then $\text{rank}(A) = m$

(c) if $\ker(A) = \{\vec{0}\}$ then $m \leq n$

pf/

(a) Note m is the # of columns. Consider the homog. linear syst $A\vec{x} = \vec{0}$.

Row reduce.

Recall the rank of A is the number of leading 1's in RREF(A).

We know $m = \text{rank}(A) = \#$ of leading 1's $= \#$ of columns.

Then each column has a leading 1, so each variable is associated to a leading 1

We also know $\ker(A)$ is the solution space to the homog. linear system $A\vec{x} = \vec{0}$.

Since each variable is associated to a leading 1, there are no "free" variables.

So $x_1 = \cdots = x_m = 0$, ie $\ker(A) = 0$.

(b) If $A\vec{x} = \vec{0}$ has only the trivial soln then each variable corresponds to a leading 1 so rank $= m$

(c) Each row has at most one leading 1, so $\boxed{\# \text{ of leading 1's is } \leq n}$

If $\ker(A) = \{\vec{0}\}$ then the lin. system $A\vec{x} = \vec{0}$ has as its only solution

$$x_1 = 0$$
$$x_2 = 0$$
$$\vdots$$
$$x_m = 0$$

$\Rightarrow$ each col. has a leading 1.

There are m columns, so in this case

$$\underbrace{m}_{\# \text{ cols}} = \# \text{ leading 1's} \leq \underbrace{n}_{\# \text{ rows}}$$

Corollary (contrapositive)

if $m > n$, ie if $A = \begin{bmatrix} & & & \\ & & & \end{bmatrix}$ then there have to be free variables

so $\ker(A)$ is not only $\vec{0}$.

Now go to § 3.2: Subspaces of $\mathbb{R}^n$ - Bases + linear independence

Def Let W be a subset of $\mathbb{R}^n$. Then W is a (linear) subspace if

(a) W contains $\vec{0}$ (the 0-vector in $\mathbb{R}^n$)

(b) W is closed under add'n of vectors

(c) W is closed under scalar mult.

Remarks

1. If you know that $W \neq \emptyset$ then (a) is a consequence of either (b) or (c)

   so it isn't really necessary to include it in the def'n

2. Conditions (b) and (c) could be replaced by

$$\text{If } \vec{w}_1, \vec{w}_2 \in W \text{ and } a_1, a_2 \in \mathbb{R} \text{ then } a_1\vec{w}_1 + a_2\vec{w}_2 \in W. \quad (*)$$

   I'll leave it to you to check:

   - if you know (*) then (b) and (c) both hold
   - if you know (b) and (c) then (*) holds.

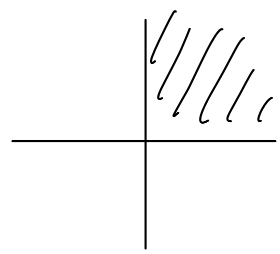
Example Last class and today we saw:

if $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transf. then

image(T) is a subspace of $\mathbb{R}^n$

and ker(T) " " " " $\mathbb{R}^m$

Ex in $\mathbb{R}^2$, let W be the 1st quadrant



Is W a subspace of $\mathbb{R}^2$?

No! (b) holds but not (c).